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ON THE ILLUMINATION OF UNBOUNDED CLOSED CONVEX SETS

BY

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ABSTRACT

In this note we prove that the illumination of an almost bounded closed convex set by minimum number of affine subspaces of given dimension can be reduced to the illumination of a bounded closed convex set of lower dimension.

1. Introduction

Let $\mathbf{K}$ be a closed convex set of the d–dimensional Euclidean space $\mathbf{E}^d$ with non-empty interior, where $d \geq 1$. We say that the affine subspace $L \subset \mathbf{E}^d \setminus \mathbf{K}$ of dimension $0 \leq \dim L \leq d-1$ illuminates the boundary point P of $\mathbf{K}$ if and only if there exists a point Q of L which illuminates P i.e. the ray emanating from P having direction vector $\overrightarrow{QP}$ intersects the interior of $\mathbf{K}$. Furthermore, we say that the affine subspaces $L_1, L_2, \ldots, L_n \subset \mathbf{E}^d \setminus \mathbf{K}$ illuminate $\mathbf{K}$ if and only if every boundary point of $\mathbf{K}$ is illuminated by at least one of the affine subspaces $L_1, L_2, \ldots, L_n$. Finally, let $I_l(\mathbf{K})$ be the smallest number of affine subspaces of dimension l lying in $\mathbf{E}^d \setminus \mathbf{K}$ which illuminate $\mathbf{K} \subsetneq \mathbf{E}^d$, where $0 \leq l \leq d-1$. Obviously, $1 \leq I_{d-1}(\mathbf{K}) \leq I_{d-2}(\mathbf{K}) \leq \cdots \leq I_0(\mathbf{K})$. The following notion was introduced in [1]. A d–dimensional closed convex set $\mathbf{K} \subsetneq \mathbf{E}^d$ is called almost bounded if and only if there exists a d–dimensional ball of $\mathbf{E}^d$ which intersects every supporting hyperplane of $\mathbf{K}$. Thus, the intersection of finitely many closed half-spaces of $\mathbf{E}^d$ is almost bounded while rotating a parabola about the axis and

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taking the convex hull of it in $\mathbf{E}^d$ we get a d−dimensional closed convex set which is not almost bounded. Clearly, there are many more examples of both types. It is proved in [1] that $I_0(\mathbf{K})$ is finite if and only if $\mathbf{K}$ is almost bounded. An equivalent condition was given in [5]. In this note we generalize this result in the following way. If $\mathbf{K} \subsetneq \mathbf{E}^d$ is almost bounded, then let $\mathbf{C}$ denote the closed convex cone which is the union of closed half-lines emanating from an interior point say, O of $\mathbf{K}$ and lying in $\mathbf{K}$. Moreover, let $Pr_L : \mathbf{E}^d \to L$ denote the orthogonal projection of $\mathbf{E}^d$ onto the affine subspace $O \in L$ which is the orthogonal complement of the affine hull aff $\mathbf{C}$ of $\mathbf{C}$ in $\mathbf{E}^d$ and let $I_l[\mathrm{cl}(Pr_L(\mathbf{K}))]$ denote the corresponding illumination number of the closure $\mathrm{cl}(Pr_L(\mathbf{K}))$ of $Pr_L(\mathbf{K})$ in L, where $0 \le l \le d-1$. Obviously, if dim $L \le l$, then we take $I_l[\mathrm{cl}(Pr_L(\mathbf{K}))] = 1$. We prove the following

THEOREM: *Let $\mathbf{K} \subsetneq \mathbf{E}^d$ be a d−dimensional almost bounded closed convex set and let $0 \le l \le d-1$. Then $Pr_L(\mathbf{K})$ is bounded and $I_l(\mathbf{K}) \le I_l[\mathrm{cl}(Pr_L(\mathbf{K}))] < +\infty$.*

If $I_0(\mathbf{K}) < +\infty$ for a d−dimensional closed convex set $\mathbf{K} \subsetneq \mathbf{E}^d$, then the d−dimensional ball containing finitely many points of $\mathbf{E}^d \setminus \mathbf{K}$ which illuminate $\mathbf{K}$ intersects every supporting hyperplane of $\mathbf{K}$. Thus, our Theorem implies the following well-known statement (see [1] and [5]).

COROLLARY 1: *Let $\mathbf{K} \subsetneq \mathbf{E}^d$ be a d−dimensional closed convex set, where $d \ge 1$. Then $I_0(\mathbf{K})$ is finite if and only if $\mathbf{K}$ is almost bounded.*

It is a very natural but still open problem to characterize all d−dimensional closed convex sets $\mathbf{K} \subsetneq \mathbf{E}^d$ for which $I_l(\mathbf{K}) < +\infty$, with some $1 \le l \le d-2$.

Boltjanskii [4] observed that $I_0(\mathbf{B}) = d+1$ for any smooth compact convex set $\mathbf{B} \subset \mathbf{E}^d$ with non-empty interior. Recently, the author [3] showed that if $\mathbf{B} \subset \mathbf{E}^d$ is a smooth compact convex set with non-empty interior, then $I_l(\mathbf{B}) = \left\lfloor \frac{d - \lceil \frac{d}{l+1} \rceil}{l} \right\rfloor + 1 = \left\lceil \frac{d+1}{l+1} \right\rceil$, where $1 \le l \le d-1$. These statements and our Theorem imply

COROLLARY 2: *Let $\mathbf{K} \subsetneq \mathbf{E}^d$ be a d−dimensional almost bounded smooth closed convex set and let $0 \le l \le d-1$. Then $I_l(\mathbf{K}) \le \left\lceil \frac{d+1}{l+1} \right\rceil$.*

Hadwiger [7], [8] conjectured that any compact convex subset of $\mathbf{E}^d$ with non-empty interior can be covered by 2^d smaller homothetic copies. This conjecture has stimulated a lot of research in geometry (see [1]). The conjecture is proved for $d = 2$ (see [1], [4], [6] and [10]) but it is unsolved for $d \geq 3$. Boltjanskii [4] and Soltan [12] showed that Hadwiger's conjecture is equivalent to the conjecture that $I_0(\mathbf{B}) \leq 2^d$ for any compact convex subset $\mathbf{B}$ of $\mathbf{E}^d$ with non-empty interior. In [1] and [2] another formulation of this problem is given. Namely, if $\mathbf{B}$ is a compact convex subset of $\mathbf{E}^d$ that contains the origin O as an interior point, then $I_0(\mathbf{B})$ is the smallest number of hyperplanes of $\mathbf{E}^d$ which strictly separate O from the faces of the polar body $\mathbf{B}^* = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \leq 1 \text{ for all } Y \in \mathbf{B}\}$, where $d \geq 1$ and $\langle\ ,\ \rangle$ denotes the usual inner product of $\mathbf{E}^d$. See also the Lemma below for a generalization of this statement. [2] proves Hadwiger's conjecture for convex polyhedra with affine symmetry. In fact, [2] contains the following more general result. If $\mathbf{P}$ is a convex d-polytope of $\mathbf{E}^d$ with affine symmetry, then $I_{d-3}(\mathbf{P}) \leq 8$ and $I_{d-2}(\mathbf{P}) = 2$, where $d \geq 3$. Hence, this and our Theorem imply

COROLLARY 3: *If $\mathbf{P} \subsetneqq \mathbf{E}^d$ is a d-dimensional convex polyhedral set (i.e. $\mathbf{P}$ is a d-dimensional intersection of finitely many closed half-spaces of $\mathbf{E}^d$) with affine symmetry, then $I_{d-3}(\mathbf{P}) \leq 8$ and $I_{d-2}(\mathbf{P}) \leq 2$ for $d = 3, 4$.*

2. Proof of Theorem

The following statement is a more general version of Lemma 6, 7 and 8 in [11].

PROPOSITION: *Let $\mathbf{K} \subset \mathbf{E}^d$ be a closed convex set that contains the origin O and let $\mathcal{F} \neq \emptyset$ be the set of all faces of $\mathbf{K}$ which do not contain O, where $d \geq 1$. Then the polar set $\mathbf{K}^* = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \leq 1 \text{ for all } Y \in \mathbf{K}\}$ is a closed convex set of $\mathbf{E}^d$ with $O \in \mathbf{K}^*$. If $\mathcal{F}^*$ denotes the set of all faces of $\mathbf{K}^*$ which are disjoint from O, then the map*

$$* : \mathcal{F} \to \mathcal{F}^*$$
$$F \mapsto F^* = \{X \in \mathbf{K}^* | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle = 1 \text{ for all } Y \in F\}$$

is a one-to-one map between $\mathcal{F}$ and $\mathcal{F}^*$ and it is inclusion reversing.

Proof: First, we prove that F^* is a face of $\mathbf{K}^*$ with $O \notin F^*$. Since $F \in \mathcal{F}$ is a face of $\mathbf{K}$ with $O \notin F$ therefore there exists a supporting hyperplane $H = \{Y \in \mathbf{E}^d | \langle \overrightarrow{OY}, \overrightarrow{OX_0} \rangle = 1\}$ of $\mathbf{K}$ with $H \cap \mathbf{K} = F$ and $O \in \mathbf{K} \subset H^+ = \{Y \in \mathbf{E}^d | \langle \overrightarrow{OY}, \overrightarrow{OX_0} \rangle \leq 1\}$. Consequently, $X_o \in F^*$, i.e. $F^* \neq \emptyset$. Now let Y_o be a relative interior point of F i.e. $Y_o \in \text{rel int } F$. Then $\mathbf{H} = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OY_0} \rangle = 1\}$ is a supporting hyperplane of $\mathbf{K}^*$ because $\mathbf{K}^* \subset \mathbf{H}^+ = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OY_0} \rangle \leq$

$1\}$ and $(\emptyset \neq) F^* \subset F' = \mathbf{H} \cap \mathbf{K}^*$. We prove that also $F^* \supset F'$ which then implies $F^* = F'$ finishing the proof of the fact that F^* is a face of $\mathbf{K}^*$ with $O \notin F^*$. Suppose that there exists $X_o \in F' \setminus F^*$. Then we have a point $Y_1 \in F$ such that $\langle \overrightarrow{OX_0}, \overrightarrow{OY_1} \rangle < 1$. Since $Y_1 \neq Y_o$ and $Y_o \in \text{rel int } F$ therefore there exists a point $Y_2 \in F$ with $\overrightarrow{OY_0} = \lambda \cdot \overrightarrow{OY_1} + (1-\lambda) \cdot \overrightarrow{OY_2}$,$0 < \lambda < 1$. But $\langle \overrightarrow{OX_0}, \overrightarrow{OY_2} \rangle \leq 1$ consequently, $\langle \overrightarrow{OX_0}, \overrightarrow{OY_0} \rangle = \lambda \cdot \langle \overrightarrow{OX_0}, \overrightarrow{OY_1} \rangle + (1-\lambda) \cdot \langle \overrightarrow{OX_0}, \overrightarrow{OY_2} \rangle < \lambda + 1 - \lambda = 1$, a contradiction.

Secondly, we observe that $(\mathbf{K}^*)^* = \mathbf{K}$. If Y is an arbitrary point of $\mathbf{K}$, then $\langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \leq 1$ for all $X \in \mathbf{K}^*$. Hence, $\mathbf{K} \subset (\mathbf{K}^*)^*$. We prove that $\mathbf{K} \supset (\mathbf{K}^*)^*$. Let $Y_o \in \mathbf{E}^d \setminus \mathbf{K}$. So there exists a hyperplane $H = \{Y \in \mathbf{E}^d | \langle \overrightarrow{OY}, \overrightarrow{OX_0} \rangle = 1\}$ which strictly separates Y_o from $(O \in)\mathbf{K}$, i.e. $\langle \overrightarrow{OY_o}, \overrightarrow{OX_0} \rangle > 1$ and $\langle \overrightarrow{OY}, \overrightarrow{OX_0} \rangle < 1$ for all $Y \in \mathbf{K}$. But then $X_o \in \mathbf{K}^*$ and so $Y_o \in \mathbf{E}^d \setminus (\mathbf{K}^*)^*$.

We finish the proof of Proposition showing that $(F^*)^* = F$ for any face $F \in \mathcal{F}$. We know that

$$\begin{aligned}(F^*)^* &= \{Y \in (\mathbf{K}^*)^* | \langle \overrightarrow{OY}, \overrightarrow{OX} \rangle = 1 \text{ for all } X \in F^*\} \\ &= \{Y \in K | \langle \overrightarrow{OY}, \overrightarrow{OX} \rangle = 1 \text{ for all } X \in F^*\} \supset F.\end{aligned}$$

We have to show that $(F^*)^* \subset F$. We have seen above that $F = H \cap \mathbf{K}$ with $H = \{Y \in \mathbf{E}^d | \langle \overrightarrow{OY}, \overrightarrow{OX_0} \rangle = 1\}$ and $\mathbf{K} \subset H^+ = \{Y \in \mathbf{E}^d | \langle \overrightarrow{OY}, \overrightarrow{OX_0} \rangle \leq 1\}$. Hence, $X_o \in F^*$. So if $Y_o \in \mathbf{K} \setminus F$, then $\langle \overrightarrow{OY_0}, \overrightarrow{OX_0} \rangle < 1$ i.e. $Y_o \in \mathbf{K} \setminus (F^*)^*$. ∎

Having proved the above Proposition we can prove the following Lemma which is the cornerstone of the proof of our Theorem. Also, it is a slight generalization of the Separation Lemma of [2]. We need the following notation. If $O \notin L$ is an affine subspace of $\mathbf{E}^d$ with $0 \leq \dim L \leq d-1$, then $\hat{L} = \cap_{Q \in L} \{H_Q | H_Q = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OQ} \rangle = 1\}\}$ is an affine subspace of dimension $\dim \hat{L} = d - \dim L - 1$ with $O \notin \hat{L}$. Finally, let $\hat{L}' = \text{cl}\{X \in \mathbf{E}^d | \overrightarrow{OX} = \overrightarrow{OY} + \lambda \cdot \overrightarrow{YO} \text{ with } Y \in \hat{L} \text{ and } \lambda \geq 0\}$.

LEMMA: *Let $\mathbf{K}$ be a closed convex set of $\mathbf{E}^d$ that contains the origin O as an interior point and let F_m be the smallest dimensional face of $\mathbf{K}$ which contains the boundary point P of $\mathbf{K}$, where $d \geq 1$. Then the affine subspace $L \subset \mathbf{E}^d \setminus \mathbf{K}$ of dimension $0 \leq \dim L \leq d-1$ illuminates P if and only if $\hat{L}' \cap F_m^* = \emptyset$ saying in that case that $\hat{L}$ co-illuminates the face $F_m^* = \{X \in \mathbf{K}^* | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle = 1 \text{ for all } Y \in F_m\}$ of the polar set $\mathbf{K}^* = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \leq 1 \text{ for all } Y \in \mathbf{K}\}$. Furthermore, $I_l(\mathbf{K}) = n$ if and only if n is the smallest integer such that there* *exist affine subspaces $\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_n$ of $\mathbf{E}^d$ of dimension $d-l-1$ with the property that every face of the polar set $\mathbf{K}^*$ which is disjoint from O can be co-illuminated by at least one of the affine subspaces $\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_n$, where $0 \le l \le d-1$.*

Proof: The Proposition implies that the map $*\colon \mathcal{F} \to \mathcal{F}^*$, $F \mapsto F^* = \{X \in \mathbf{K}^* | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle = 1 \text{ for all } Y \in F\}$ is a one–to–one map between $\mathcal{F}$ and $\mathcal{F}^*$ and it is inclusion reversing.

Let $P(F,\text{resp.})$ be a boundary point (face, resp.) of $\mathbf{K}$. Then we define the following closed convex cones:

$\mathbf{C}_P = \bigcap\{H^+ | H^+ \text{ is a supporting half-space to } \mathbf{K} \text{ bounded by } H \text{ with } P \in H\}$,
$\mathbf{C}_F = \bigcap\{H^+ | H^+ \text{ is a supporting half-space to } \mathbf{K} \text{ bounded by } H \text{ with } F \subset H\}$,
$\overline{\mathbf{C}}_F = \overrightarrow{PO} + \mathbf{C}_F$ with any $P \in \operatorname{aff} F$ and
$\mathbf{C}_F^* = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \le 0 \text{ for all } Y \in \overline{C}_F\}$ called polar cone of $\overline{C}_F$.

It is easy to prove that if F is a face of $\mathbf{K}$, then

$$\mathbf{C}_F^* = \{X \in \mathbf{E}^d |\ \overrightarrow{OX} = \lambda \cdot \overrightarrow{OY} \text{ with } \lambda \ge 0 \text{ and } \langle \overrightarrow{OY}, \overrightarrow{OZ} \rangle \le 1$$
$$\text{for all } Z \in \mathbf{K} \text{ and } \langle \overrightarrow{OY}, \overrightarrow{OZ_0} \rangle = 1 \text{ for all } Z_0 \in F\}.$$

Thus, $\mathbf{C}_F^* = \operatorname{pos} F^*$, where pos(.) denotes the positive hull of a set.

Let F_m be the smallest dimensional face of $\mathbf{K}$ which contains the boundary point P of $\mathbf{K}$. The affine subspace $L \subset \mathbf{E}^d \setminus \mathbf{K}$ of dimension l illuminates P if and only if there exists $Q \in L$ such that the open ray $r^P_{\overrightarrow{QP}}$ emanating from P having direction vector $\overrightarrow{QP}$ lies in the interior $\operatorname{int} \mathbf{C}_P$ of $\mathbf{C}_P$ i.e. $r^P_{\overrightarrow{QP}} \subset \operatorname{int} \mathbf{C}_{F_m}$. Then $r^P_{\overrightarrow{QP}} \subset \operatorname{int} \mathbf{C}_{F_m}$ if and only if $\langle \overrightarrow{OY}, \overrightarrow{PQ} \rangle > 0$ for any $Y (\ne O) \in \mathbf{C}^*_{F_m} = \operatorname{pos} F^*_m$. As $\langle \overrightarrow{OY}, \overrightarrow{PQ} \rangle > 0$ for any $Y (\ne O) \in \operatorname{pos} F^*_m$ if and only if $\langle \overrightarrow{OY}, \overrightarrow{PQ} \rangle > 0$ for any $Y \in F^*_m$ we get that the affine subspace L illuminates P if and only if there exists $Q \in L$ such that $\langle \overrightarrow{OY}, \overrightarrow{OQ} \rangle > \langle \overrightarrow{OY}, \overrightarrow{OP} \rangle = 1$ for any $Y \in F^*_m$. Thus, L illuminates P if and only if there exists $Q \in L$ such that the hyperplane $H_Q = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OQ} \rangle = 1\} \supset \hat{L}$ strictly separates O from the face $F^*_m = \{X \in \mathbf{K}^* | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle = 1 \text{ for all } Y \in F_m\}$ of the polar set $\mathbf{K}^*$. Finally, this is true if and only if $\hat{L}' \cap F^*_m = \emptyset$ i.e. $\hat{L}$ co-illuminates the face F^*_m of $\mathbf{K}^*$.

As the map $*\colon \mathcal{F} \to \mathcal{F}^*$, $F \mapsto F^* = \{X \in \mathbf{K}^* | \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle = 1 \text{ for all } Y \in F\}$ is a one–to–one map having the above argument we get immediately that the affine subspaces $L_1, L_2, \ldots, L_n \subset \mathbf{E}^d \setminus \mathbf{K}$ of dimension l illuminate $\mathbf{K}$ if and only if every face in $\mathcal{F}^*$ of the polar set $\mathbf{K}^*$ can be co-illuminated by at least one of the affine subspaces $\hat{L}_i = \bigcap_{Q \in L_i} \{H_Q = \{X \in \mathbf{E}^d | \langle \overrightarrow{OX}, \overrightarrow{OQ} \rangle = 1\}\}$ $i = 1, 2, \ldots, n$ of dimension $d - l - 1$. This completes the proof of the lemma. ∎

The proof of our Theorem relies on the Lemma. We assume that $\mathbf{K} \subsetneq \mathbf{E}^d$ is an almost bounded closed convex set that contains the origin O as an interior point. Thus, the Proposition yields that $\mathbf{K}^* \subset \mathbf{E}^d$ is a compact convex set with $O \in \mathbf{K}^*$ and since $\mathbf{K}$ is almost bounded dist $(\cup\mathcal{F}^*, O) > 0$. Let F_m^* be the smallest dimensional face of $\mathbf{K}^*$ which contains O. F_m^* can be identical to the improper face $\mathbf{K}^*$.

We prove that $O \in$ rel int F_m^* using induction on the dimension $d^*(\geq 1)$ of $\mathbf{K}^*$. If $d^* = 1$ or $d^* = 2$, then it is easy to see that $O \in$ rel int F_m^*. So suppose that the claim is true for any d'-dimensional compact convex set whose faces not containing O are bounded away from O and take a d^*-dimensional compact convex set $\mathbf{K}^* \subset \mathbf{E}^{d^*}$ with O lying on the boundary bd $\mathbf{K}^*$ of $\mathbf{K}^*$ and with dist $(\cup\mathcal{F}^*, O) > 0$, where $2 \leq d' < d^*$. Let

$$\mathbf{C}^* = \bigcup\{r_{\overrightarrow{OX}}^O \mid r_{\overrightarrow{OX}}^O \text{ denotes the closed ray emanating from } O \text{ having direction vector } \overrightarrow{OX} \text{ with } (O \neq) X \in \mathbf{K}^*\}.$$

It is obvious that dist $(\cup\mathcal{F}^*, O) > 0$ if and only if there exists a d^*-dimensional closed ball $B^{d^*}(O, \varepsilon) \subset \mathbf{E}^{d^*}$ centered at O with radius $\varepsilon > 0$ such that $\mathbf{K}^* \cap B^{d^*}(O, \varepsilon) = \mathbf{C}^* \cap B^{d^*}(O, \varepsilon)$. Let $H_m \subset \mathbf{E}^{d^*}$ be the supporting hyperplane of $\mathbf{K}^*$ for which $H_m \cap \mathbf{K}^* = F_m^*$. Since in case of $O \in$ rel int F_m^* we are done we suppose that $O \in$ rel bd $F_m^* = F_m^* \setminus$ rel int F_m^*. Consequently, dim $F_m^* \geq 1$. As dim $F_m^* <$ dim $\mathbf{K}^* = d^*$ and $F_m^* \cap B^{d^*}(O, \varepsilon) = (\mathbf{C}^* \cap H_m) \cap B^{d^*}(O, \varepsilon)$ i.e. the union of the faces of F_m^* which are disjoint from O lies at distance $\geq \varepsilon$ from O we get by induction that F_m^* possesses a face $\overline{F}_m^*$ with $O \in$ rel int $\overline{F}_m^*$ (Fig. 1). Hence, there exists a (d^*-2)-dimensional affine subspace $\overline{H}_m$ which supports F_m^* in H_m such that $\overline{H}_m \cap F_m^* = \overline{H}_m \cap \mathbf{K}^* = \overline{F}_m^*$. Let $(O \in)\overline{H}_m^\perp$ be the 2-dimensional affine subspace of $\mathbf{E}^{d^*}$ which is totally orthogonal to $\overline{H}_m$ and let $Pr(\mathbf{K}^*)$ be the orthogonal projection of $\mathbf{K}^*$ onto $\overline{H}_m^\perp$ parallel to $\overline{H}_m$. Obviously, $Pr(\mathbf{K}^*)$ is a convex domain whose boundary contains O and $Pr(\mathbf{K}^*) \cap B^{d^*}(O, \varepsilon) = Pr(\mathbf{C}^*) \cap B^{d^*}(O, \varepsilon)$ i.e. every face of $Pr(\mathbf{K}^*)$ which is disjoint from O lies at distance $\geq \varepsilon$ from O. Consequently, by induction there exists a face $Pr(F_m^\perp)$ of $Pr(\mathbf{K}^*)$ with $O \in$ rel int $Pr(F_m^\perp)$. Hence, for the face $F_m^\perp$ of $\mathbf{K}^*$ whose orthogonal projection onto $\overline{H}_m^\perp$ is $Pr(F_m^\perp)$ we have that $O \in$ rel int $F_m^\perp$ and $F_m^* \neq F_m^\perp$, a contradiction. Thus, $O \in$ rel int F_m^*.

It is easy to show that the cone $\mathbf{C}^*$ defined above is the polar cone of $\mathbf{C}$. Thus,

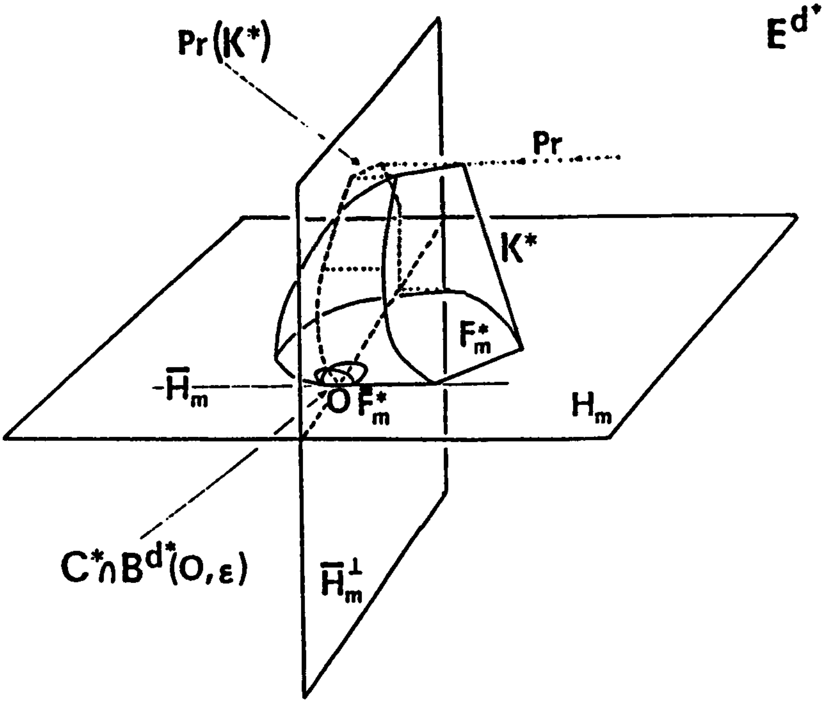

Fig. 1

we have

$$\begin{aligned}
\mathbf{K}^* &= \{X \in \mathbf{E}^d \mid \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \le 1 \text{ for all } Y \in \mathbf{K}\} \\
&= \{X \in \mathbf{E}^d \mid \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \le 1 \text{ for all } Y \in \text{bd } \mathbf{K}\} \\
&\quad \bigcap \{X \in \mathbf{E}^d \mid \langle \overrightarrow{OX}, \overrightarrow{OZ} \rangle \le 0 \text{ for all } Z \in \mathbf{C}\} \\
&= \{X \in \mathbf{E}^d \mid \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \le 1 \text{ for all } Y \in \text{bd } \mathbf{K}\} \cap \mathbf{C}^* \text{ and} \\
[Pr_L(\mathbf{K})]^{*L} &= \{X \in L \mid \langle \overrightarrow{OX}, \overrightarrow{OPr_L(Y)} \rangle \le 1 \text{ for all } Y \in \mathbf{K}\} \\
&= \{X \in L \mid \langle \overrightarrow{OX}, \overrightarrow{OPr_L(Y)} \rangle \le 1 \text{ for all } Y \in \text{ bd } \mathbf{K}\} \\
&= \{X \in L \mid \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \le 1 \text{ for all } Y \in \text{bd } \mathbf{K}\} \\
&= L \cap \{X \in \mathbf{E}^d \mid \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \le 1 \text{ for all } Y \in \text{bd } \mathbf{K}\}.
\end{aligned}$$

Then

$$\begin{aligned}
[Pr_L(\mathbf{K})]^{*L} \cap \mathbf{C}^* &= L \cap (\{X \in \mathbf{E}^d \mid \langle \overrightarrow{OX}, \overrightarrow{OY} \rangle \le 1 \text{ for all } Y \in \text{bd } \mathbf{K}\} \cap \mathbf{C}^*) \\
&= L \cap \mathbf{K}^*.
\end{aligned}$$

Since L is totally orthogonal to aff $\mathbf{C}$ therefore $L \subset \mathbf{C}^*$ and so $[Pr_L(\mathbf{K})]^{*L} = [Pr_L(\mathbf{K})]^{*L} \cap \mathbf{C}^* = L \cap \mathbf{K}^*$. We have proved that $O \in \operatorname{rel\,int} F_m^*$. This implies that aff $F_m^* = L$ from which we get that

$$[\mathrm{cl}(Pr_L(\mathbf{K}))]^{*L} = [Pr_L(\mathbf{K})]^{*L} = L \cap \mathbf{K}^* = F_m^*.$$

As a partial result we have got that $Pr_L(\mathbf{K})$ is bounded. The Lemma implies that $I_l[\mathrm{cl}(Pr_L(\mathbf{K}))]$ is the smallest integer n such that there exist affine subspaces $\hat{L}_1, \hat{L}_2, \dots, \hat{L}_n$ of L of dimension dim $L - l - 1$ with the property that every face of the polar set F_m^* can be co-illuminated by at least one of the affine subspaces $\hat{L}_1, \hat{L}_2, \dots, \hat{L}_n$, where $0 \le l \le \dim L - 1$. We distinguish Case 1: $F_m^* = \mathbf{K}^*$ and Case 2: F_m^* is a face of dimension $\le d^* - 1$ of $\mathbf{K}^*$, where dim $\mathbf{K}^* = d^* \ge 1$.

CASE 1: Either $\mathbf{K} = \mathrm{cl}(Pr_L(\mathbf{K}))$ or $\mathbf{K}$ is a cylinder with base $\mathrm{cl}(Pr_L(\mathbf{K}))$. Thus, it is obvious that $I_l(\mathbf{K}) \le I_l[\mathrm{cl}(Pr_L(\mathbf{K}))]$. Finally, as $\mathrm{cl}(Pr_L(\mathbf{K}))$ is compact it is sufficient to recall the known fact that

$$I_l[\mathrm{cl}(Pr_L(\mathbf{K}))] \le I_0[\mathrm{cl}(Pr_L(\mathbf{K}))] < +\infty$$

(see [1] or [4]).

CASE 2: F_m^* is a face of dimension $\le (d^* - 1)$ of the d^*-dimensional compact convex set $\mathbf{K}^* \subset \mathbf{E}^{d^*}$ with $O \in \operatorname{rel\,int} F_m^*$ and dist $(\cup\mathcal{F}^*, O) > 0$. We have seen that $[\mathrm{cl}(Pr_L(\mathbf{K}))]^{*L} = F_m^*$, where $O \in L = \operatorname{aff} F_m^*$ is the orthogonal complement of aff $\mathbf{C}$ in $\mathbf{E}^d$. Since F_m^* is bounded therefore Case 1 and the Lemma imply that there are affine subspaces $H_1(m), H_2(m), \dots, H_n(m)$ of L of dimension dim $L - l - 1$ with the property that $n = I_l[\mathrm{cl}(Pr_L(\mathbf{K}))]$ and every face of the polar set F_m^* can be co-illuminated by at least one of the affine subspaces $H_1(m), H_2(m), \dots, H_n(m)$, where $0 \le l \le \dim L - 1$. Let H (H^+, resp.) be the supporting hyperplane (supporting half-space bounded by H, resp.) of $\mathbf{K}^*$ in $\mathbf{E}^{d^*}$ with $H \cap \mathbf{K}^* = F_m^*$. Let H_i be the affine subspace of $\mathbf{E}^{d^*}$ of dimension $d^* - l - 1$ orthogonal to aff F_m^* with $H_i \cap \operatorname{aff} F_m^* = H_i(m)$, where $1 \le i \le n$. Finally, let $B^{d^*}(O, R)$ be a d^*-dimensional closed ball of $\mathbf{E}^{d^*}$ centered at O with radius $R > 0$ such that int $B^{d^*}(O, R) \supset \mathbf{K}^*$. There are many ways to rotate H_i about $H_i(m)$ toward O. We choose the following. Let h be the affine function which is positive on H^+ and zero on H and which satisfies $|h(P)| = 1$ for points P lying at distance 1 from H. For $i = 1, \dots, n$ let $H_i[\epsilon] = \{X \in \mathbf{E}^{d^*} \mid \overrightarrow{OX} = \overrightarrow{OQ} + \epsilon \cdot \overrightarrow{QP} + h(P) \cdot \overrightarrow{QO} \text{ with } P \in H_i\}$ of dimension

$d^* - l - 1$, where Q is a point in $H_i(m)$ and $\epsilon > 0$. One can easily verify that $H_i[\epsilon]' \cap H^+ \cap B^{d^*}(O,R)$ tends to $H_i(m)' \cap B^{d^*}(O,R)$ in the Hausdorff metric, as ϵ tends to 0. (The notation $'$ is the same as in the Lemma.)

Now the claim is that for some $\epsilon > 0$ and for every face F^* of $\mathbf{K}^*$ disjoint from O one of the affine subspaces $H_i[\epsilon]$ co-illuminates F^*, where $1 \le i \le n$. If not, then there is a sequence of faces $F^*(k)$ of $\mathbf{K}^*$ which are disjoint from O and $F^*(k)$ intersects each of the sets $H_i[\frac{1}{k}]'$, where $1 \le i \le n$. The Blaschke selection theorem ([9], pp. 98) implies that a subsequence of the sequence $F^*(k)$ will converge. Say the limit is M. M cannot contain O, since the faces of $\mathbf{K}^*$ which do not contain O are bounded away from O. Because O is a relative interior point of F^*_m, it follows that the same holds for every relative interior point of F^*_m, and therefore M does not intersect the relative interior of F^*_m. As M is convex, this shows that $M \cap F^*_m$ is contained in a proper face , say F^*, of F^*_m. But M must intersect each of the Hausdorff limits of the sequences $H_i[\frac{1}{k}]' \cap H^+ \cap B^{d^*}(O,R)$. These Hausdorff limits are just $H_i(m)' \cap B^{d^*}(O,R)$ and one gets a contradiction, since that implies that F^* is not co-illuminated by any of the affine subspaces $H_i(m)$.

This completes the proof of the theorem. ∎

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